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Strongly edge triangle regular graphs and a conjecture of Kotzig

B. Radhakrishnan Nair, A. Vijayakumar*

Department of Mathematics, Cochin University of Science and Technology, Cochin 682 022, India

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Abstract

The concepts of strongly vertex triangle regular graphs and strongly edge triangle regular graphs are introduced. An expression for the triangle number of a vertex in the composition of two graphs is obtained. It is proved that a self-complementary graph is strongly regular if and only if it is strongly edge triangle regular. Using these, we continue the analysis of a conjecture of Kotzig on self-complementary graphs.

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. By $G(p,q)$ we mean a graph G of order p and size q. The symbols $V(G), E(G), d(u), N(u)$, $E(u), \bar{G}$ and $\langle S \rangle$ denote, respectively, the vertex set of G, the edge set, the degree of a vertex u, the set of vertices adjacent to u, the set of edges incident at u, the complement of G and the subgraph of G induced by $S \subset V(G)$. Notations and definitions not specified here are from [4].

The number of triangles in G containing a vertex u is called the triangle number of u in G, denoted by $t(u)$, and $\bar{t}(u)$ will denote the triangle number of u in $\bar{G}$. Triangle number $t(e)$ of an edge is also defined in similar terms. An expression for $t(u) + \bar{t}(u)$ is given in [6]. In a graph G, two vertices (edges) are similar if there is an automorphism of G that maps one of these vertices (edges) onto the other. A graph is vertex-symmetric (edge-symmetric) if every pair of its vertices (edges) are similar.

A graph G is self-complementary if G and $\bar{G}$ are isomorphic. A self-complementary graph will be of order $4k$ or $4k+1$ for some natural number k and of diameter 2 or 3. If it is regular, then its order is $4k+1$ and diameter is 2. For a self-complementary graph G there exists isomorphism $\sigma : G \to \bar{G}$, which are permutations of $V(G)$, called

* Corresponding author.

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complementing permutations. The set of all complementing permutations of G is denoted by $\mathscr{C}(G)$. A vertex u in a self-complementary graph is called a fixed vertex if $\sigma(u) = u$ for some complementing permutation σ. Ringel [8] and Sachs [9] independently proved the existence of exactly one fixed vertex associated with each $\sigma \in \mathscr{C}(G)$ when G is of odd order and non-existence otherwise. $F(G)$ denotes the set of all fixed vertices in a self-complementary graph G and $\hat{F}(G)$ the set of all vertices with traiangle number $k(k-1)$ in a regular self-complementary graph of order $4k+1$. In [5], Kotzig has conjectured that $F(G) = \hat{F}(G)$ for a regular self-complementary graph. The conjecture holds trivially for the graph of order 5. Rao [7] characterised $F(G)$, proved that $F(G) \subseteq \hat{F}(G)$ and disproved the conjecture for $p = 9$. We have characterised $\hat{F}(G)$ in [6].

In this paper, we introduce the concepts of vertex triangle regular, edge triangle regular, strongly vertex triangle regular (s.v.t.r) and strongly edge triangle regular (s.e.t.r) graphs and obtain an expression for the triangle number of a vertex in the composition of two graphs. It is proved that a graph G is strongly regular if and only if both G and $\bar{G}$ are s.e.t.r and deduce (i) Lemma 4.3 of Rao in [7], (ii) the well-known relation $(p-r-1)\mu = r(r-\lambda-1)$ connecting the parameters of strongly-regular graph [3] and that (iii) a self-complementary graph is strongly regular if and only if it is s.e.t.r. Finally, we discuss more about Kotzig's conjecture on regular self-complementary graphs in continuation of our earlier work.

2. Triangle number

In [4], the composition $G(H)$ of two graphs G and H is defined as the graph with $V(G) \times V(H)$ as vertex set and (u_1, v_1) and (u_2, v_2) are adjacent if either u_1 and u_2 are adjacent in G or $u_1 = u_2$ and v_1 and v_2 are adjacent in H. It is known [4] that $G(H)$ and $H(G)$ need not be isomorphic, $G(H)$ is connected if and only if G is connected and $G(H)$ is regular if and only if both G and H are so. Further, if G and H are vertex-symmetric (self-complementary) then $G(H)$ is also vertex-symmetric (self-complementary) [7].

Remark. The composition $G(H)$ can be obtained by replacing each vertex u_i of G by a copy of H and each edge u_iu_j of G by all the possible edges between the copies of H corresponding to u_i and u_j.

Theorem 2.1. *Let $G(p_1, q_1)$ and $H(p_2, q_2)$ be two graphs. Then the triangle number of any vertex (u, v) in $G(H)$ is given by*

$$t(u,v) = t(v) + q_2 d(u) + p_2 d(u) d(v) + p_2^2 t(u).$$

Proof. The triangles at (u, v) in $G(H)$ are formed in the following ways only.

(i) A triangle at v in H is also a triangle at (u, v) in $G(H)$. The number of such triangles at (u, v) is $t(v)$.

(ii) An edge in a copy of H corresponding to a neighbour of u in G forms a triangle at (u,v). The number of such triangles is $q_2d(u)$.

(iii) Each edge of H at v forms a triangle in $G(H)$ with each of the vertices in the copy of H that corresponds to a neighbour of u in G. This contributes $p_2d(u)d(v)$ to $t(u,v)$.

(iv) Each triangle in G at u contributes p_2^2 triangles in $G(H)$ at (u,v). The number of triangles so formed is $p_2^2t(u)$.

Hence the expression for $t(u,v)$. □

Corollary 2.2. *If there are t_1 triangles in $G(p_1,q_1)$ and t_2 triangles in $H(p_2,q_2)$, then the number of triangles in $G(H)$ is*

$$p_1t_2 + p_2^3t_1 + 2p_2q_1q_2.$$

Theorem 2.3. *Let G and H be graphs of order p_1 and p_2. The triangle number of an edge e in $G(H)$ joining the vertices (u_1,v_1) and (u_2,v_2) is given by*

$$t(e) = \begin{cases} t(e_1) + p_2d(u_1) & \text{if } u_1 = u_2, e_1 = v_1v_2 \in E(H), \\ p_2t(e_2) + d(v_1) + d(v_2) & \text{if } u_1 \neq u_2, e_2 = u_1u_2 \in E(G). \end{cases}$$

The proof is on similar lines to that of Theorem 2.1.

Definition. A graph G is vertex triangle regular if all of its vertices have the same triangle number and is strongly vertex triangle regular (s.v.t.r) if it is regular also.

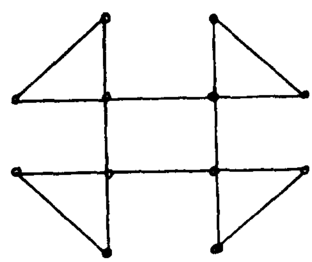

Fig. 1. A vertex triangle regular graph which is not s.v.t.r.

Theorem 2.4. *If G and H are strongly vertex triangle regular graphs, then so is $G(H)$.*

Proof. The vertices in a s.v.t.r. graph have same degree and same triangle number. Clearly, all vertices in $G(H)$ also have the same degree — namely $d(H)+d(G)\cdot p(H)$, and by Theorem 2.1 same triangle number. □

Definition. A graph G is edge triangle regular if all of its edges have the same triangle number and is strongly edge triangle regular (s.e.t.r) if it is regular also.

Lemma 2.5. *Every strongly edge triangle regular graph is strongly vertex triangle regular.*

Proof. Let G be s.e.t.r., $d(u) = r$ and $t(e) = 1$ for every vertex u and edge e. Then, $t(u) = \frac{1}{2}\sum_{e \in E(u)} t(e) = \frac{1}{2}rt$ for every $u \in V(G)$. Thus, G is s.v.t.r. □

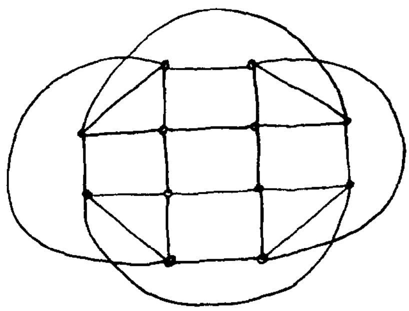

Fig. 2. A s.v.t.r. graph which is not s.e.t.r.

Definition (Buckley and Harary [2], and Cameron [3]). A graph G is strongly regular with parameters p, r, λ and μ if it is of order p, regular of degree r, any two adjacent vertices have precisely λ common neighbours and any two non-adjacent vertices have precisely μ common neighbours.

Lemma 2.6 (Cameron [3]). *If G is strongly regular with parameters p, r, λ and μ, then $\bar{G}$ is also strongly regular and the parameters are $p, p - r - 1, p - 2r + \mu - 2$ and $p - 2r + \lambda$.*

Theorem 2.7. *A graph G is strongly regular if and only if both G and $\bar{G}$ are strongly edge triangle regular.*

Proof. Let G be a strongly regular path with parameters p, r, λ and μ. Then $\bar{G}$ is strongly regular with parameters $p, p - r - 1, p - 2r + \mu - 2$ and $p - 2r + \lambda$. Hence, in G, $d(u) = r$ and $t(e) = \lambda$ for every vertex u and edge e, and in $\bar{G}$, $d(u) = p - r - 1$ and $t(e) = p - 2r + \mu - 2$ for every vertex u and edge e. Thus, G and $\bar{G}$ are s.e.t.r.

Conversely, let G and $\bar{G}$ be s.e.t.r. and let $d(u) = r$ for every vertex u in G, $t(e) = t$ for every edge in G and $t(e) = \bar{t}$ for every edge in $\bar{G}$. Then in G, any two adjacent vertices have t common neighbours and any two non-adjacent vertices have $2r + \bar{t} - p + 2$ common neighbours. So G is strongly regular with parameters p, r, t and $2r + \bar{t} - p + 2$. □

Corollary 2.8 (Cameron [3]). *If p, r, λ and μ are parameters of a strongly regular graph, then*

$$(p - r - 1)\mu = r(r - \lambda - 1). \tag{2.1}$$

Proof. For every vertex u,

$$t(u) = \frac{1}{2} \sum_{e \in E(u)} t(e) = \tfrac{1}{2} r\lambda \tag{2.2}$$

and

$$\bar{t}(u) = \tfrac{1}{2}(p - r - 1)(p - 2r + \mu - 2). \tag{2.3}$$

But we have [6, Corollary 2.5]

$$t(u) + \bar{t}(u) = \binom{p-1}{2} - \tfrac{3}{2} r(p - r - 1). \tag{2.4}$$

Substituting for the L.H.S. of (2.4) from (2.2) and (2.3) we get (2.1). □

Theorem 2.9. *A self-complementary graph is strongly edge triangle regular if and only if it is strongly regular with parameters $4k + 1$, $2k$, $k - 1$ and k for some natural number k.*

Proof. Let G be a self-complementary graph. Then the equivalence of s.e.t.r. and strongly regular properties follows by Theorem 2.7, since G and $\bar{G}$ are isomorphic.

In both cases, G is regular and hence $p = 4k + 1$ and $r = 2k$ for some natural number k. We have [6, Corollary 2.6] $t(u) + \bar{t}(u) = 2k(k - 1)$ for every vertex u. By Lemma 2.5 and isomorphism of G and $\bar{G}, t(u) = t(u) = k(k - 1)$. Hence, $\lambda = k - 1$ by (2.2). Now $\mu = k$ follows from (2.1).

Corollary 2.10 (Rao [7]). *If G is an edge-symmetric self-complementary graph, then G is strongly regular with parameters $4k + 1$, $2k$, $k - 1$ and k for some natural number k.*

Proof. Let G be an edge-symmetric self-complementary graph. Then G is s.e.t.r. and the result follows. □

3. More about a conjecture of Kotzig

Kotzig has conjectured that $F(G) = \hat{F}(G)$ for a regular self-complementary graph G of order $p = 4k + 1$, where

$$F(G) = \{u \in V(G):\ \sigma(u) = u \text{ for some } \sigma \in \mathscr{C}(G)\}$$

and

$$\hat{F}(G) = \{u \in V(G):\ t(u) = k(k-1)\}.$$

Rao [7] proved that $F(G) = V(G)$ if and only if G is vertex-symmetric, $F(G) \subseteq \hat{F}(G)$ and gave counterexamples to disprove the equality. But in [6], we have observed that there is a fallacy in the construction of examples except for $k = 2$ (the corresponding graph is denoted by G_9). Consequently, the conjecture was open for regular self-complementary graph of order $p = 4k + 1$, $k \geqslant 3$. In pursuance of this conjecture we have.

Lemma 3.1. *For a regular self-complementary graph G of order $p = 4k + 1$, $\hat{F}(G) = V(G)$ if and only if G is strongly vertex triangle regular.*

Proof. Let G be a s.v.t.r. self-complementary graph of order $p = 4k + 1$. Clearly, $\hat{F}(G) \subseteq V(G)$. Let u be a fixed vertex. Then $t(u) = k(k-1)$. Since G is s.v.t.r, $t(v) = t(u)$ for any vertex $v \in V(G)$. So, $t(v) = k(k-1)$ and hence $v \in \hat{F}(G)$. Conversely, $F(G) = V(G)$ implies that $t(v) = k(k-1)$ for every vertex $v \in V(G)$ and so G is s.v.t.r. $\square$

So, s.v.t.r. self-complementary graphs, which are not vertex-symmetric, will give counterexamples to the conjecture. Rao's graph G_9 (Fig. 3) also turns out to be such a graph. We shall now construct such graphs of order 17 and 33 using the circulant graph.

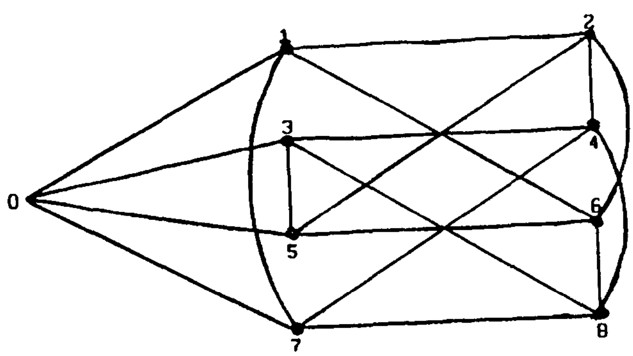

Fig. 3.

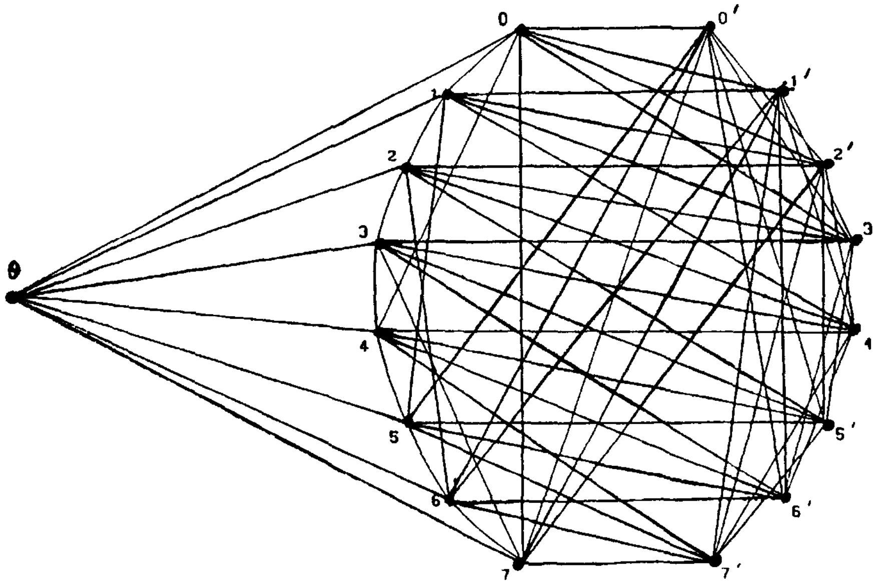

Fig. 4.

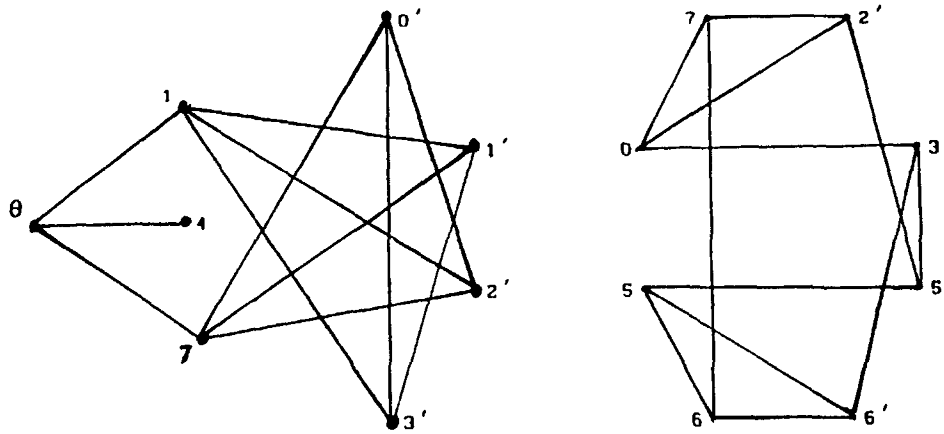

Fig. 5.

Definition (*Boesch* [1], *and Buckley and Harary* [2]). For a given positive integer p, let $a_1, a_2, \ldots, a_n$ be integers such that $0 < a_1 < a_2 < \cdots < a_n < (p+1)/2$. Then the circulant graph $C(p; a_1, a_2, \ldots, a_n)$ is the graph on p vertices $0, 1, 2, \ldots, p-1$ with vertex i adjacent to each vertex $i \pm a_j \pmod{p}$.

Construction of G_{17}. Take a single vertex θ, a copy of the circulant graph $C(8; 1, 4)$ with vertices labelled as $0, 1, 2, \ldots, 7$ and a copy of its complement $C(8; 2, 3)$ with

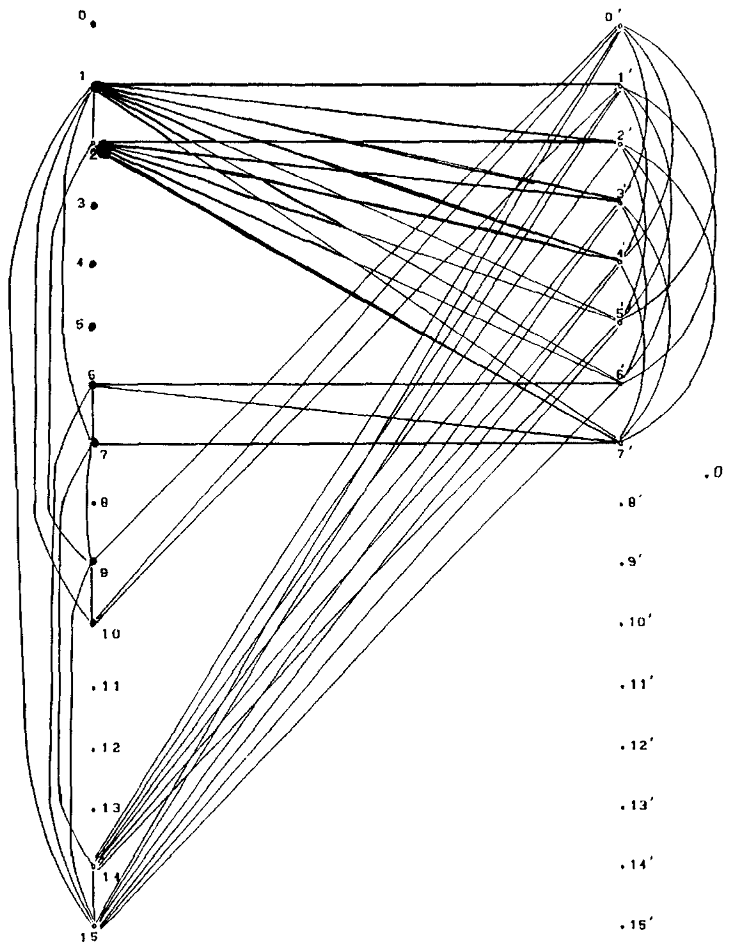

Fig. 6.

vertices labelled as $0', 1', 2', \ldots, 7'$. Join each vertex i to θ, i', $i'+1$, $i'+2$ and $i'+3$, addition being taken modulo 8 and $i'+j$ means $(i+j)'$. The graph G_{17} (Fig. 4) so obtained is self-complementary, (θ) $(00'11'22'\cdots 77')$ is a complementing permutation. From the diagram of $G_{17'}$ its strong vertex triangle regularity is clear. But, from Fig. 5 it follows that 0 and $0'$ are non-similar vertices and hence G_{17} is not vertex-symmetric.

Construction of G_{33}. Take a single vertex θ, a copy of the circulant graph $C(16; 1, 2, 6, 7)$ with vertices labelled as $0, 1, 2, \ldots, 15$ and a copy of its complement with vertices labelled as $0', 1', 2', \ldots, 15'$. Join each vertex i to i', $i'+1$, $i'+7, \ldots,$ (mod 16) and each i' to θ. The resulting graph G_{33} is self-complementary under the complementing permutation (θ) $(00'11'22'\cdots 1515')$ and is s.v.t.r. But $\langle N(\theta)\rangle \cong C(16; 3, 4, 5, 8)$ and $\langle N(0)\rangle$ is shown in Fig. 6. Hence, G_{33} is not vertex-symmetric.

Thus by Theorem 2.4, the composition of G_9, G_{17} and G_{33} will give counterexamples of order $p = 9^{r_1}17^{r_2}33^{r_3}$, where r_1, r_2, r_3 are non-negative integers, not all zero. Attempts to find counter examples for more values of p are in progress.

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